The Homotopy of Topological Graphs Based on Khalimsky Arcs

Rueiher Tsaur*†, Chun-Hong Huang*‡
* Department of Computer Information and Network Engineering
College of Electrical Engineering and Computer Science
Lunghwa University of Science and Technology, Taoyuan County, Taiwan
† rueiher@mail.lhu.edu.tw
‡ ch.huang@mail.lhu.edu.tw

***Abstract*—In this paper, we aim to develop a suitable homotopy theory of finite topological graphs by Khalimsky arcs. The notions developed are considered for the investigation of the algebraic invariants of topological graphs for their topological and graphical classifications.**

***Keywords*—Connected ordered topological spaces, digital topology, homotopy, Khalimsky spaces, topological graphs.**

I. Introduction

The Khalimsky space is an alternative topological approach to digital topology introduced by Khalimsky [1], [2], aiming to utilize the standard topological theory and avoid the need to work with different adjacency relations (on the same digital space) in Rosenfeld's graph-theoretic model of the digital spaces. Another approach to Khalimsky spaces was independently introduced by Kovalevsky [4], who considered the Khalimsky spaces as structures consisting of elements of different dimensions, in terms of cellular complexes.

In [1] and [2], Khalimsky has developed the homotopy of Khalimsky spaces by using self-defined digital lines. The homotopy he defined is similar to the corresponding homotopy in algebraic topology, but instead of the unit interval $[0, 1]$ by taking digital lines with arbitrary finite or infinite points. In [8], Pultr has developed the homotopy of general graphs by finite graphs (I_k, R_k), where $I_k = \{1, 2, \ldots, k\}$ and $R_k = \{(i, i+1), (i, i), (i+1, i) | 1 \leq i \leq k\}$.

In the present paper we aim to develop a suitable homotopy theory of finite topological graphs [9], [10], [11] by Khalimsky arcs. The definitions defined below are considered for the investigation of the algebraic invariants of topological graphs for their topological and graphical classifications. This work is a precursor of approaching the general homotopy theory, fundamental groups, and fixed point theorems of topological graphs.

II. Topological Graphs and Continuum

In order to relate topology with graphs, as proposed by Smyth in [9], [10], a topological space may be considered as the limit of an inverse sequence of finite graphs. The graphs in such a sequence are considered as increasingly better discrete representations of the space, and that the sequence as a whole approximates the space is justified mathematically by the fact that the space may be derived from its limit via a quotienting operation. Thus a formal framework, topological graphs, is developed as a generalized topological structure within which we can handle standard topological spaces and general graphs. In short, topological graphs are structures:

1) Which embody topology as well as a binary relation, thereby constituting a generalization of ordered topological spaces to accommodate standard topological spaces and general graphs. A topological graph is a general graph if its topology is discrete, and a standard topological space if its binary relation is identity.
2) Such that the category **TopGr** provides a suitable framework in which various spaces may be approximated by inverse sequence of finite graphs. Where the finite graphs in the sequence are considered as increasingly better finite representations of the spaces, and the infinite spaces may be derived from the limit of inverse sequence of finite graphs in a canonical way via a quotient operation.
3) Suitable for computational (approximation) purpose, as the closure of binary relations provides the constraint for finite observations. A closed subset of a space is the extension property which holds if and only if a certain finitely observable event does *not* occur. Thus any two unrelated points are observable in a topological graph with closed binary relation.
4) Which may have further contribution to geometry, since the concept of observable properties of points of topological graphs in some senses is similar to the concept of indistinguishability in tolerance geometry.

A *topological graph* (X, T, R) is a structure such that T is a topology, R is a closed reflexive binary relation satisfying $(\forall u, v \in T, x \in u \;\&\; R(u) \subseteq v \Rightarrow y \in v) \Rightarrow xRy$. A *graph morphism* from the topological graph (X, T_X, R_X) to the topological graph (Y, T_Y, R_Y) is a continuous function w.r.t. the topologies T_X and T_Y, which is relation preserving. For a topological graph $G = (X, T, R)$, we say G is compact Hausdorff if (X, T) is compact Hausdorff. Moreover, G is said to be connected if for every partition $\{u, v\}$ of X into T-open sets, there exist $x \in u, y \in v$ such that xRy or yRx. For more information about topological graphs, please refer [9], [10], [11] and [12].

Note 2.1: Without any further notice, the topological graphs are finite; the maps between two topological graphs are graph morphisms.

Definition 2.2: [11] A *continuum* is a compact connected topological graph.

In the case of identity relation, a continuum becomes classical continuum in general topology (e.g. [5]). Moreover, every finite connected reflexive topological graph with discrete topology is a continuum. Hence the topological graphical definition of continuum contains classical continuum as well as general finite connected graphs.

As both compactness and connectedness are graph-morphism invariants, any graph-morphic image of a continuum is a continuum, hence the inverse sequence of continua is well-defined and the limit of an inverse sequence of continua is a continuum ([12], Proposition 3.8).

A point x is said to be a *cut point* of the connected topological graph G if $G \setminus \{x\}$ is separated into R-disjoint open sets. The cut point x separates the points $a, b, \ldots \in G$ if there exists a partition $\{U_a, U_b, \ldots\}$ of $G \setminus \{x\}$ into R-disjoint open sets such that $a \in U_a, b \in U_b, \ldots$. In the case that $G \setminus \{x\}$ is separated into exactly two R-disjoint open sets, we call x an *arc-like* cut point (or strong cut point) of the connected topological graph G.

From the point of view of investigating the linear structures for topological graphs, it is convenient to pick up the idea of *connected ordered topological space* (*COTS*) [3]. We say a connected topological graph G satisfying the *COTS* condition if and only if for any three distinct points $x, y, z \in G$, one of $\{x, y, z\}$ separates the other two. It is easy to show that a connected topological graph G satisfying the *COTS* condition has at most two non-cut points (end points), and all the cut points are arc-like cut points.

Definition 2.3: [11] A *linear* topological graph is a connected locally connected (read as usual) topological graph satisfying the *COTS* condition.

We are especially interested in a restricted form of linear topological graphs, which we can simulate the general "line". We have the definition so-called *arc* defined as followed:

Definition 2.4: [11] An *arc* is a second countable continuum satisfying the *COTS* condition.

From Definition 2.4, clearly an arc contains exactly two end points (non-cut points) because of its compactness ([3], Lemma 2.11), and all the cut points are arc-like cut points.

III. Khalimsky Arcs

The so-called *Khalimsky integer line* is defined to be the set of all integers $\mathbb{Z}$ with its natural order together with its interval alternating topology T_{ia}, where T_{ia} is a topology defined by sets of types $(-\infty, 2i], [2j, \infty), i, j \in \mathbb{Z}$ (or alternately $(-\infty, 2i-1], [2j-1, \infty), i, j \in \mathbb{Z}$) as a subbasis. The *Khalimsky digital line* then can be defined as a finite connected subspace of the Khalimsky integer line. Fig.1 shows a portion of a standard Khalimsky digital line.

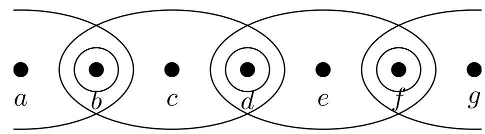

Fig. 1. Khalimsky digital line

Clearly a Khalimsky digital line is a connected ordered topological space [3]. Hence it immediately implies that a connected subset of a Khalimsky digital line is again a Khalimsky digital line.

Definition 3.1: Let $B \subseteq A$ be a subset of a Khalimsky digital line A. We call B a *sub-Khalimsky digital line* of A if B itself is a Khalimsky digital line. We denote $B \trianglelefteq A$ if B is a sub-Khalimsky digital line of A.

Theorem 3.2: [3] A connected ordered topological space contains at most two end points and every other point is a arc-like cut point.

The proofs of the following lemmas are trivial:

Lemma 3.3: Let A be a Khalimsky digital line and $B \trianglelefteq A$. If $a \in A$, then $(B \cup \{a\}) \trianglelefteq A$ if and only $a \in B$ or $B \cup \{a\}$ is connected.

Lemma 3.4: Let A be a Khalimsky digital line with two end points a and b, and $B \trianglelefteq A$ such that $a \in B$ or $b \in B$ (but not both). Then $A \setminus B \trianglelefteq A$.

Definition 3.5: (*separation condition for a connected ordered topological space*) If $\prec$ is a linear ordering on a set X and $x \in X$, we denote $L(x) = \{y \in X \mid y \prec x\}$ and $U(x) = \{y \in X \mid x \prec y\}$. Let A be a connected ordered topological space, we say a linear ordering $\prec$ satisfying the *separation condition* if and only if $L(x)$ and $U(x)$ are the partition of $A \setminus \{x\}$ for all $x \in A$.

The following theorem provides an alternative approach to Theorem 2.7 in [3], it claims that there are only two nature linear orderings satisfying the separation condition for a Khalimsky digital line.

Theorem 3.6: Let A be a Khalimsky digital line with two end points a and b, then there are only two natural linear orderings $\prec$ and $\succ$ ($=\prec^{-1}$) such that we can embed into A. For $c, d \in A$, we say $c \prec d$ if and only if there are sub-Khalimsky lines C, D with $a \in C, D$ such that $c \in C, d \in D$ implies $C \subset D$. Conversely, we have $c \succ d$ if and only if there are sub-Khalimsky digital lines C, D with $b \in C, D$ such that $c \in C$ and $d \in D$ implies $D \subset C$.

Proof: Clearly the definitions of $\prec$ and $\succ$ are well-defined since A is a Khalimsky digital line. From Lemmas 3.3 and 3.4, it is obvious that $c \prec d$ iff $c \succ d$. The conditions of the partial orderings of $\prec$ and $\succ$ can be easily shown by the definitions of $\prec$ and $\succ$. We show $\prec$ and $\succ$ are linear orderings, and it is sufficient to show $\prec$ is a linear ordering as $\prec = \succ^{-1}$: For any $c, d \in A$, if $c = d$, then nothing need to be proved. If $c \neq d$, by Lemma 3.3 (since the singleton set $\{a\}$ and $\{b\}$ are Khalimsky digital lines), we can find a sub-Khalimsky digital line $C \trianglelefteq A$ such that $a \in C$. WLOG, we can assume $c \in C$ and $d \notin C$, hence $c \prec d$ as $C \subset A$. The "only" condition is similar to the proof of Theorem 2.7 in [3]. ∎

Lemma 3.7: The linear orderings $\prec$ and $\succ$ satisfy the separation condition.

Let A be a Khalimsky digital line with two end points a and b. For convenient, we may say that we travel A from the point a to the point b if $a \prec b$, and we denote it by ${}_aA_b$; otherwise we denote it by ${}_bA_a$ iff $a \succ b$. Sometimes we may call a an initial point and b an final point of ${}_aA_b$.

For any point c of a Khalimsky digital line ${}_aA_b$, we denote the successor (resp. predecessor) of c by $c+$ (resp. $c-$) the immediate adjacent point of c such that $c- \prec c \prec c+$ when $c \notin \{a, b\}$ and $c- \equiv a$ iff $c \equiv a$ & $c+ \equiv b$ iff $c \equiv b$. We also denote $c0+ = c, c1+ = c+$, and $c2+ = (c++) = (c1+)+ = (c+)+, \ldots$, etc., and $c0- = c, c1- = c-$, and $c2- = (c--) = (c1-)- = (c-)-, \ldots$, etc..

Theorem 3.8: Let A be a Khalimsky digital line with at least three points. Then A has two end points which are both closed or open (but not both) w.r.t. the topology of *COTS* if and only if $(\sharp A \mod 2) = 1$, where $\sharp A$ is the numbers of points of A.

Proof: Each point of the connected ordered topological space X is closed or open but not both if X has at least three points [3]. Therefore we have each point of the Khalimsky digital line A is closed or open (but not both).

($\Rightarrow$): We claim $(\sharp A \mod 2) = 1$ if the two end points of A are both closed or open (but not both). Suppose $(\sharp A \mod 2) = 0$, then A has even numbers of points. We assume A has n points, where n is even. WLOG, we may assume $A \equiv {}_{a_1}A_{a_n}$ such that both a_1 and a_n are closed (open). By Lemma 2.8(c) of [3], x is closed iff y is open for two adjacent points $x, y \in A$. Therefore we have $a_n = a_1(n-1)+$ is open (closed) as n is even, which contradicts the hypothesis that two end points of A are both closed or open.

($\Leftarrow$): To show that two end points of A are both closed or open if $(\sharp A \mod 2) = 1$ is trivial (similar to the proof of ($\Rightarrow$)). ■

In the reminder of section we focus our attention on the so-called *Khalimsky arcs*, which we will be able to simulate the general "intervals" to develop our homotopy theory of topological graphs, also further to investigate the algebraic invariant of topological graphs.

Notice that a Khalimsky digital line is not necessarily an arc by regarding it as a topological space, i.e., with identity relation; since it is not compact if it contains infinite points. With the specialization pre-ordering embedded into a Khalimsky digital line, regarded as its relation, we can convey a Khalimsky digital line into a directed graph. As for a good example we can convey the Khalimsky digital line in Fig. 1 into a directed graph in Fig. 2.

Definition 3.9: We call $G = (B, T, R)$ is a Khalimsky arc if B is the set of all points of a Khalimsky digital line A, T is the interval topology induced by the linear ordering $\prec$ (or $\succ$) of A, and R is the specialization pre-ordering $\leq$ induced from the *COTS* topology of A.

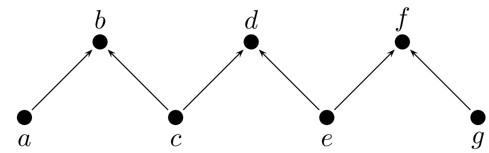

Fig. 2. Khalimsky arc

IV. Homotopy of Topological Graphs by Khalimsky Arcs

We shall use the same terminology and notations, such as end points (initial and final points), successor (predecessor) of a point, etc., of Khalimsky digital lines, for Khalimsky arcs, if there is no confusion occurred.

Definition 4.1: Let A be a Khalimsky arc and $c \in A$. We say c is a *minimal point* of A iff $c \leq c-$ & $c \leq c+$, and c is a maximal point iff $c- \leq c$ & $c+ \leq c$.

It is obvious that $c+$ and $c-$ (if exists) are minimal (maximal) if and only if c is maximal (minimal) when c is any point of a Khalimsky arc. Furthermore, we have the following lemma that is directly derived from Theorem 3.8:

Lemma 4.2: Let A be a Khalimsky arc with at least three points. Then A has two end points which are both minimal or maximal (but not both) if and only if $(\sharp A \mod 2) = 1$.

Let $\mathfrak{S}$ be the collection of all Khalimsky arcs A which satisfy the condition $(\sharp A \mod 2) = 1$. By Lemma 4.2, A is a Khalimsky arc with two end points which are both minimal or maximal (or a singleton Khalimsky arc, which is a Khalimsky arc containing a single point). Therefore it is clear that $\mathfrak{S}$ can be classified into two classes whose intersection is a set containing all singleton Khalimsky arcs. One class denoted by $\mathfrak{p}\mathfrak{S}$ contains all elements of $\mathfrak{S}$ with two minimal end points (and all singleton Khalimsky arcs); the other class denoted by $\mathfrak{c}\mathfrak{S}$ contains all elements of $\mathfrak{S}$ with two maximal end points (and all singleton Khalimsky arcs).

Definition 4.3: Let $A, B \in \mathfrak{p}\mathfrak{S}$ such that $\sharp A = n$ with two end points a and a', and $\sharp B = m$ with two end points b and b'. We define $A \triangledown B \in \mathfrak{p}\mathfrak{S}$ ($A \triangledown B \in \mathfrak{p}\mathfrak{S}$ can be easily proved by Lemma 4.2) to be a Khalimsky arc with $n+m-1$ points such that the first $n-1$ points of $A \triangledown B$ are considered as points of A and the last $m-1$ points of $A \triangledown B$ are considered as points of B. And the topology of $A \triangledown B$ is the interval topology of the points of $A \triangledown B$, where the concatenation point of $A \triangledown B$ is one of $\{a \triangledown b, a \triangledown b', a' \triangledown b, a' \triangledown b'\}$ which is depended on the initial and final points of A and B. The operator is called *lexicographic union* or *oriented union*.

Clearly $A \triangledown B$ defined in Definition 4.3 is well-defined as the concatenation point of $A \triangledown B$ is always minimal. For example, the Fig 4 shows the lexicographic union of two Khalimsky arcs $A, B \in \mathfrak{p}\mathfrak{S}$ in Fig 3. Moreover, we can extend Definition 4.3 into any arbitrary finite lexicographic union of Khalimsky arcs in $\mathfrak{p}\mathfrak{S}$. Also it is easy to check Definition 4.3 is still valid if we replace $\mathfrak{p}\mathfrak{S}$ by $\mathfrak{c}\mathfrak{S}$.

Definition 4.4: Let $A \in \mathfrak{p}\mathfrak{S}, B \in \mathfrak{c}\mathfrak{S}$ be two Khalimsky arcs, and G is a topological graph. We call the maps $f : A \to$

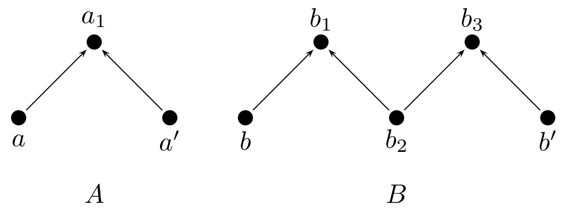

Fig. 3. $A, B \in \mathfrak{p}\mathfrak{S}$

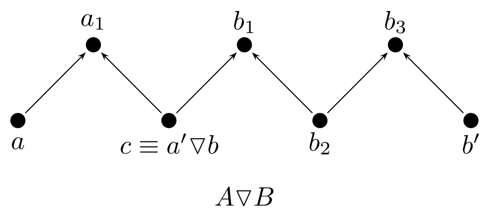

Fig. 4. $A \nabla B \in \mathfrak{p}\mathfrak{S}$

G and $g : B \to G$ a $\mathfrak{p}$-path and a $\mathfrak{c}$-path of G respectively.

Definition 4.5: $A \in \mathfrak{S}$ is a Khalimsky arc with two end points a and b, and G is a topological graph. A map $f : A \to G$ is said to be a *directed* path w.r.t. $(a \prec b)$ of G if $f(a(2i)+) = f(a(2i \text{ minus } 1)+)$[1] or $f(a(2i)+) = f(a(2i \text{ plus } 1)+)$[2] for all $i \in \mathbb{Z}^+ \cup \{0\}$. The operators *minus* and *plus* are the normal subtraction and addition.

Theorem 4.6: Let $G_1 = (X, T_X, R_X)$ and $G_2 = (Y, T_Y, R_Y)$ be topological graphs. Then $(X \times Y, T_{X\times Y}, R_{X\times Y})$ is a topological graph, where $T_{X\times Y}$ is the product topology of T_X and T_Y, $R_{X\times Y} = \{((x,y),(x',y')) \mid (x,x') \in R_X \ \&\ (y,y') \in R_Y\}$.

Proof: We claim $R_{X\times Y}(A \times A') = R_X(A) \times R_Y(A')$, where A, B and A', B' are subsets of X and Y respectively. Clearly $R_{X\times Y}(A \times A') = \{(x,x') \mid (a,a')R_{X\times Y}(x,x') \text{ for some } (a,a') \in A \times A'\} = \{(x,x') \mid aR_Xx \ \&\ a'R_Yx' \text{ for some } (a,a') \in A \times A'\} = \{x \mid aR_Xx \text{ for some } a \in A\} \times \{x' \mid a'R_Yx' \text{ for some } a' \in A'\} = R_X(A) \times R_Y(A')$.

1) $R_{X\times Y}$ is closed if R_X and R_Y are closed: R_X is closed in $X \times X$ and R_Y is closed in $Y \times Y$, hence $R_X(X)$ is closed in X and $R_Y(Y)$ is closed in Y. So we have $R_{X\times Y}(X \times Y) = R_X(X) \times R_Y(Y)$ is closed in $X \times Y$, which shows $R_{X\times Y}$ is closed.
2) $R_{X\times Y}$ is reflexive if R_X and R_Y are reflexive: Clearly we have $(x,y)R_{X\times Y}(x,y)$ as $xR_Xx \ \&\ yR_Yy$ by the reflexivity of R_X and R_Y for all $x \in X, y \in Y$.
3) $[(\forall u, v \in T_{X\times Y}, (x,y) \in u \ \&\ R_{X\times Y}(u) \subseteq v) \Rightarrow (x',y') \in v] \Rightarrow (x,y)R_{X\times Y}(x',y')$: Let $u, v \in T_{X\times Y}$, we have $u = \bigcup_i (u_i \times u'_i), v = \bigcup_j (v_j \times v'_j)$ for some u_i, v_j, and u'_i, v'_j open sets of X and Y respectively. If $((x,y) \in u \ \&\ R_{X\times Y}(u) \subseteq v$ such that $(x',y') \in v)$ is true, then there are some $u_k \in< u_i >$ and $u'_l \in< u'_i >$ with $x \in u_k, y \in u'_l$ such that $R_{X\times Y}(u_k \times u'_l) = R_X(u_k) \times R_Y(u'_l) \subseteq v \subseteq \pi_1(v) \times \pi_2(v)$ such that $x' \in \pi_1(v)$ and $y' \in \pi_2(v)$. It is trivial that $\pi_1(v)$ and $\pi_2(v)$ are open in X and Y, hence we have xR_Xx' and yR_Yy'. This shows $(x,y)R_{X\times Y}(x',y')$.

■

We denote $G_1 \times G_2 = (X \times Y, T_{X\times Y}, R_{X\times Y})$. From Theorem 4.6, it is easy to check that $T_{X\times Y}$ is Hausdorff if T_X and T_Y are Hausdorff. Also $R_{X\times Y}$ is symmetric if R_X and R_Y are symmetric, as if $(x,y)R_{X\times Y}(x',y')$ then clearly we have $xR_Xx' \ \&\ yR_Yy'$, thus $x'R_Xx \ \&\ y'R_Yy$, and which implies $(x',y')R_{X\times Y}(x,y)$.

Following homotopy theory of topological graphs is derived based on two classes of Khalimsky arcs- $\mathfrak{p}\mathfrak{S}$ and $\mathfrak{c}\mathfrak{S}$. The main reason of this restriction is due to the definition of lexicographic union of Khalimsky arcs.

Definition 4.7: Let $(X, T_X, R_X), (Y, T_Y, R_Y)$ be topological graphs, f and g are maps of (X, T_X, R_X) into (Y, T_Y, R_Y). We say f is $\mathfrak{p}$-*homotopic* to g if there exists a Khalimsky arc $A \in \mathfrak{p}\mathfrak{S}$, and a map $H : (X, T_X, R_X) \times A \to (Y, T_Y, R_Y)$ such that $H(x,a) = f(x), H(x,b) = g(x)$ for all $x \in X$, where a, b are end points of A. We write $f \approx^{\mathfrak{p}} g$ if f is $\mathfrak{p}$-homotopic to g.

Similarly, we have $\mathfrak{c}$-homotopy if we replace $A \in \mathfrak{p}\mathfrak{S}$ by $A \in \mathfrak{c}\mathfrak{S}$ in Definition 4.7. We denote $f \approx^{\mathfrak{c}} g$ if f is $\mathfrak{c}$-homotopic to g.

Definition 4.8: With notations and terminology are same as Definition 4.7, we say f is homotopic to g if f is $\mathfrak{p}$-homotopic to g or f is $\mathfrak{c}$-homotopic to g. we denote $f \approx g$ if f is homotopic to g.

Definition 4.9: $G_1 = (X, T_X, R_X), G_2 = (Y, T_Y, R_Y)$ are topological graphs, and f, g are maps of G_1 into G_2. We say f and g are $\mathfrak{p}$-homotopic ($\mathfrak{c}$-homotopic, homotopic) relative to G if there exists a Khalimsky arc $A \in \mathfrak{p}\mathfrak{S}$ ($A \in \mathfrak{c}\mathfrak{S}$, $A \in \mathfrak{S}$), and a homotopy $H : G_1 \times A \to G_2$ such that $H(w,c) = f(w) = g(w)$ for all $w \in G \ \&\ c \in A$, where G is a subgraph of G_1. The homotopy H is called a $\mathfrak{p}$-homotopy ($\mathfrak{c}$-homotopy, homotopy) relative to G and we write $f \approx^{\mathfrak{p}}_G g$ ($f \approx^{\mathfrak{c}}_G g$, $f \approx_G g$).

It is clear that we have $f \approx^{\mathfrak{p}} g \Leftrightarrow f \approx^{\mathfrak{p}}_G g$, $f \approx^{\mathfrak{c}} g \Leftrightarrow f \approx^{\mathfrak{c}}_G g$, and $f \approx g \Leftrightarrow f \approx_G g$ if and only if $G = \emptyset$.

Theorem 4.10: With notations and terminology are same as Definitions 4.7, 4.8, and 4.9, the homotopy $\approx^{\mathfrak{p}}$ ($\approx^{\mathfrak{c}}$, $\approx$) and $\approx^{\mathfrak{p}}_G$ ($\approx^{\mathfrak{c}}_G$, $\approx_G$) are equivalence relations.

Proof: WLOG, it is enough to prove $\approx^{\mathfrak{p}}_G$ is an equivalence relation only, as $\approx^{\mathfrak{p}}$ is a special case of $\approx^{\mathfrak{p}}_G$ whenever $G = \emptyset$. Also the proofs of $\approx^{\mathfrak{c}}, \approx^{\mathfrak{c}}_G$ and $\approx, \approx_G$ being equivalence relations are similar to the proof of $\approx^{\mathfrak{p}}_G$.

If $f, g, h : G_1 \to G_2$ are maps from G_1 to G_2 with $f(w) = g(w) = h(w)$ for all $w \in G$, then we have:

1) $\approx^{\mathfrak{p}}_G$ is reflexive: Given $f : G_1 \to G_2$, then clearly $H : G_1 \times A \to G_2, H(x,c) = f(x), \forall c \in A, \forall A \in \mathfrak{p}\mathfrak{S}$, is a $\mathfrak{p}$-homotopy relative to G. So we have $f \approx^{\mathfrak{p}}_G f$.

[1] If $i = 0$, we assign $f(a(\text{minus } 1)+) = f(a)$. By Lemma 4.2, there exists n such that $b = a(n)+, n \in \mathbb{Z}^+ \cup \{0\}$ and n is even. In this case we assign $f(a(2j)+) = f(a(2j \text{ minus } 1)+) = f(b)$ for all $j \geq n/2$.

[2] Similar to Footnote 1, but assign $f(a(2j)+) = f(a(2j \text{ plus } 1)+) = f(b)$ for all $j \geq n/2$.

2) $\approx_G^{\mathfrak{p}}$ is symmetric: Given $f, g : G_1 \to G_2$. If $f \approx_G^{\mathfrak{p}} g$, then there exists a Khalimsky arc $A \in \mathfrak{p}\mathfrak{S}$, and a map $H : G_1 \times A \to G_2$ such that $H(x, a) = f(x), H(x, b) = g(x)$ with $H(w, c) = f(w) = g(w)$ for all $x \in X, w \in G$ & $c \in A$, where a and b are end points of A such that $a \prec b$. Define $H' : G_1 \times \ _bA_a \to G_2$ by $FH'(x, d) = FH(x, d)$ for all $d \in A$, where F is a forgetful functor which forgets the binary relation. We claim H' is a homotopy relative to G from g to f:
For checking the continuity of H', clearly H' is continuous as $FH' = FH$. For checking the relation-preserving of H', it is clear that our definition of homotopy does not change the specialization pre-orderings $\leq$ of $_bA_a$ and $_aA_b$, hence $H' : G_1 \times_b A_a \to G_2$ is relation-preserving w.r.t. $R_{G_1 \times_b \leq_a}$ and R_{G_2}. As for an example, if $(x, c)R_{G_1 \times_b \leq_a}(y, d)$, from Theorem 4.6 we have $xR_{G_1}y$ and $c \ _b\leq_a d$, hence we have $xR_{G_1}y$ and $c \ _a\leq_b d$ (by the definition of $_bA_a$ and $_aA_b$). So we have $(x, c)R_{G_1 \times_a \leq_b}(y, d)$. H is a homotopy, i.e., H is relation-preserving, so we have $H(x, c)R_{G_2}H(y, d)$, and hence $H'(x, c)R_{G_2}H'(y, d)$ since $FH' = FH$.
For showing $H'(x, b) = g(x), H'(x, a) = f(x), \forall x \in X$, it is trivial since $H'(x, b) = H(x, b) = g(x)$ and $H'(x, a) = H(x, a) = f(x)$.
Finally, as $H(w, c) = f(w) = g(w)$ for all $w \in G$ & $c \in A$, therefore we have $H'(w, c) = g(w) = f(w)$ for all $w \in G$ & $c \in A$, which shows H' is G-relative.

3) $\approx_G^{\mathfrak{p}}$ is transitive: Given $f, g, h : G_1 \to G_2$ with $f \approx_G^{\mathfrak{p}} g$ and $g \approx_G^{\mathfrak{p}} h$, we claim $f \approx_G^{\mathfrak{p}} h$. Let $H : G_1 \times A \to G_2$ be a $\mathfrak{p}$-homotopy of f and g relative to G such that $H(x, a_0) = f(x), H(x, a_1) = g(x)$ with $H(w, a) = f(w) = g(w)$ for all $x \in X, w \in G$ & $a \in A$, and $F : G_1 \times B \to G_2$ be a $\mathfrak{p}$-homotopy of g and h relative to G such that $F(x, b_0) = g(x), F(x, b_1) = h(x)$ with $F(w, a) = g(w) = h(w)$ for all $x \in X, w \in G$ & $a \in A$. Define $K : G_1 \times C \to G_2$ by

$$K(x, c) = \begin{cases} H(x, c), & \text{if } c \in A, \\ F(x, c), & \text{if } c \in B, \end{cases}$$

where $C = A \triangledown B$. K is well-defined as if $c = a_1 \triangledown b_0$, then we have $H(x, a_1) = g(x) = F(x, b_0)$.
For showing R-preserving of K, the proof is trivial from the definition of lexicographic union and the similar proof of (2). Clearly $K(x, a_0) = H(x, a_0) = f(x)$, $K(x, b_1) = F(x, b_1) = h(x)$. Moreover, we have $K(w, c) = f(w) = h(w)$ as $K(w, c) = f(w) = g(w)$ when $c \in A$, $K(w, c) = g(w) = h(w)$ when $c \in B$ for all $w \in G$ & $c \in C$. (PS. $K(w, a_1 \triangledown b_0) = H(w, a_1) = g(w) = F(w, b_0)$).
Finally, we still need to show that K is continuous. Since $C = A \triangledown B$ is the lexicographic union of A and B, for any closed set O of G_2, we have $K^{-1}(O) = H^{-1}(O) \cup F^{-1}(O)$, where $H^{-1}(O)$ is closed in $G_1 \times A$ and $F^{-1}(O)$ is closed in $G_1 \times B$. Clearly A and B are closed in $A \triangledown B$ by the pasting lemma, hence $G_1 \times A$ and $G_1 \times B$ are closed in $G_1 \times (A \triangledown B)$. So we have $H^{-1}(O)$ and $F^{-1}(O)$ are closed in $G_1 \times (A \triangledown B)$, which shows K is continuous.

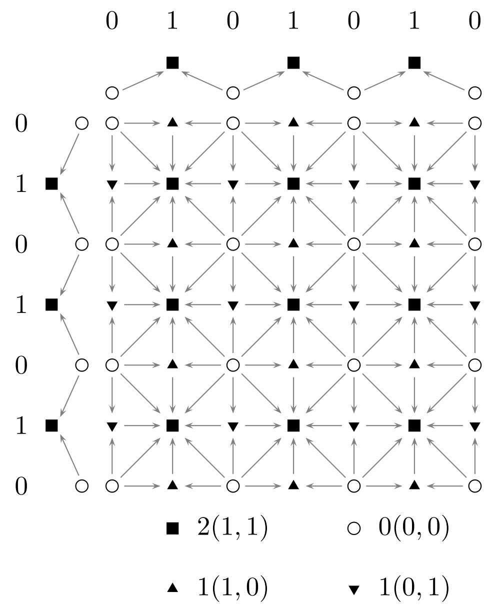

Fig. 5. A portion of a digital plane

Therefore from (1), (2), and (3), we have $\approx_G^{\mathfrak{p}}$ is an equivalence relation. ■

V. Digital Planes

Let $G_1 = (A, T_A, R_A) \in \mathfrak{p}\mathfrak{S}$ and $G_2 = (B, T_B, R_B) \in \mathfrak{p}\mathfrak{S}$, then $D = G_1 \times G_2 = (A \times B, T_{A\times B}, R_{A\times B})$ is well defined (as proved in Theorem 4.6) with respect to the topological graphical sense, and we call it a digital plane induced from Khalimsky arcs G_1 and G_2. It is easy to check the relation $R_{A\times B}$ of D is one-to-one and onto corresponding to the adjacent relation in the Khalimsky digital plane described in [3] and [6], and the topology $T_{A\times B}$ is the product interval topology. Fig. 5 shows a portion of a digital plane D.

There are four kinds of points in the digital plane D, namely $0(0, 0), 2(1, 1), 1(0, 1)$ and $1(1, 0)$, where $(0, 1)$ and $(1, 0)$ are called *mixed* points in [3] and [6] and we sometimes denotes them together as 1 since they have the same behaviour in the digital plane. As regarding with the orders of $0, 1$ and 2, clearly we have the following diagram (see Fig. 6).

It is not difficult to check that all results of the Khalimsky digital plane described in [3] can be extended easily for digital plane D with relation $R_{A\times B}$. As for a typical example, it was claimed in [3] that their proof of the Jordan Curve Theorem uses purely digital topological methods with making no appeal to the continuous version of the theorem, however we can easily prove our Jordan Curve Theorem for digital plane D with a similar approach. Notice that a Jordan curve in the digital plane D w.r.t. the notion in [3] is defined to be a connected set J with $\sharp J \geq 4$ such that $J \setminus \{j\}$ for any $j \in J$ is an arc:

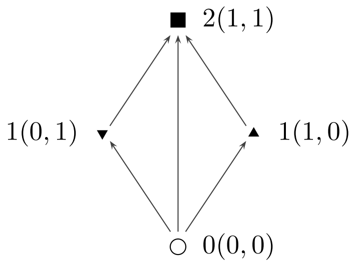

Fig. 6. Orders of $0, 1$ and 2

Theorem 5.1: If J is a Jordan curve in the digital plane D and J does not meet the border (see [3] for the definition of *border*), then $D \setminus J$ has exactly two components. The component which meets the border is called the outside, the other is called the inside.

Then, how about the topological graphical counterpart of Jordan Curve Theorem for the digital plane D? Clearly we can find that the components in Theorem 5.1 are saturated open sets and hence the theorem still works.

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