First-Order Differentiator and Integrator Design Using Bilinear Transformation

蔡林憲 吳聲威
龍華科技大學 龍華科技大學
助理教授 學生
ginggle@mail.lhu.edu.tw bcbod@yahoo.com.tw

NSC95-2221-E-262-00

Department of Electronic Engineering
Lunghwa University of Science and Technology
300 Wanshou Rd., Sec. 1, Guishan Shiang, Taoyuan 33306, Taiwan, R.O.C.

ABSTRACT

Simple and accurate formulations are employed to represent discrete-time infinite impulse response (IIR) processes of first-order differentiator and integrator. These formulations allow them to be eligible for wide-band applications. Both first-order differentiator and integrator have an almost linear phase. The new differentiator has an error of less than 1 % for the range 0-0.8π of normalized frequency and the new integrator has an error of less than 1.1 % for the range 0-0.8π of normalized frequency.

Keywords --- wide-band, linear phase

摘要

簡單和準確公式被用於代表離散時間的無限脈衝反應(IIR)在一階的微分器與積分器上，這些公式可以使用在寬頻應用，不論是微分器或積分器都有近似線性相位，新微分器有小於1％誤差範圍在0-0.8π 正規化的頻率之內。新積分器有小於1.1%誤差範圍在0-0.8π正規化的頻率之內。

關鍵字--頻寬、線性相位

1. INTRODUCTON

The differentiator is a very useful tool to determine and estimate time derivatives of a signal. It has been used extensively in many areas, such as image processing, speech systems and digital control. In radars, the velocity and acceleration of objects are computed from position measurements using differentiators [1]. The integrator is an instrumental tool to estimate the time integral of measured signals. It has been used extensively in many areas, such as coherent detection and correlation estimation. The integrator can also be employed to implement high-frequency active filters [2]. In other words, the integrator not only plays an important role in determining the inter-relation among various signals, it can also detect the history of a solo signal. In the Fourier spectral analysis, the spectral of a measured signal is the output of an integrator that takes the time integration of the multiplication of a measured signal by harmonic signals [1].

Various methods have been developed to design both discrete finite impulse response (FIR) and infinite impulse response (IIR) differentiators [3]-[11]. Al-Alaoui [3] used Simpson's rule to develop a stable second-order recursive differentiator and used the interpolation method to develop a stable minimum-phase differentiator and designed approximately linear phase IIR digital filters [4]. Tseng [5] studied a fractional order FIR differentiator by solving linear equations of Vandermonde form. In order to develop a wide band differentiator, Khan and Ohba [6] employed the central difference approximations of the derivative of a function to obtain a maximally linear differentiator. On the other hand, Trapezoidal rule and Simpson's rule in the Z domain are two popular methods used for integrators. It is known that a piecewise linear approximation leads to the Trapezoidal rule and a piecewise quadratic approximation produces the Simpson's rule. As a result, a Simpson's rule yields a more complicated formula. An important aspect of the previous investigation is that the exploration focused on the improvement of linearity over a wide frequency band. In this paper, we design a first-order and wider operating frequency bandwidth than those previously reported.

2.NEW DISCRETE-TIME DIFFERENTIATOR

It is well known that the operation of time derivative of a signal is represented by a complex-frequency variable s in the Laplace transform representation. Neglecting the loss factor, the complex-frequency variable s is equal to $j\Omega$ i.e., $s= j\Omega$, where Ω is the signal angular frequency. As a result, a differentiator is a high-pass filter and the amplitude of its system function increases linearly as the signal frequency increases. We consider a transformation relating the complex-frequency variable s and the discrete-time variable z in the z domain as follows

$$s = \frac{2}{T_d} \frac{(1 - z^{-1})}{(1 + dz^{-1})} \qquad (1)$$

where T_d is a normalization constant, d is a real constant and z^{-1} represents a unit of time delay. Physically, T_d is the sampling time interval in the DSP study. If d is set equal to 1, the transformation in (1) is called a bilinear transformation, which is widely used in converting analog prototypes to discrete-time prototypes. When the frequency response of the differentiator is concerned, the parameter z in (1) is replaced with the following relation.

$$z = e^{j\omega} \qquad (2)$$

Where ω is the frequency angle and $0 \le \omega \le \pi$. The value of d strongly affects the linearity of the transformation in (1). On the other hand, the multiplication constant $2/T_d$ dictates the amplitude response of (1). It is required that the amplitude response of (1) should be less than unity for the normalized frequencies. Fig.1 shows the relative errors of the amplitude responses of (1) as a function ω for d=0 to 1, with 0.2 increments. The relative error of the amplitude response of the ideal differentiator is also shown in Fig. 1.

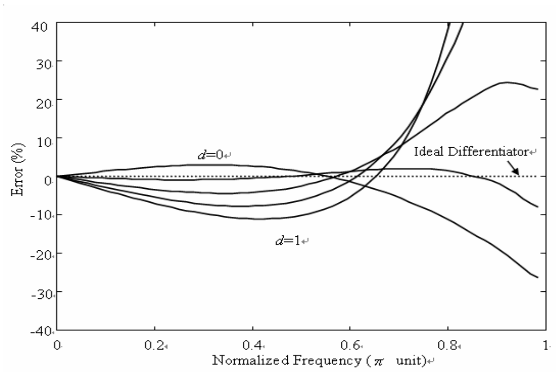

Fig. 1 The relative errors of the amplitude response for d=0 to 1, with 0.2 increments and the ideal differentiator.

If d is 0.2, the relative error is less than 1 % for the range 0-0.5 π of normalized frequency and 2 % for the range 0-0.8 π of normalized frequency. Therefore, the value of d of the transformation in (1) is proper to be selected as the system function of a wide-band differentiator. Fig. 2 shows the amplitude responses of (1) for different values of d when they are compared to an ideal differentiator. The ideal differentiator is assumed to have precisely linear amplitude response for all frequencies, as shown in Fig. 2.

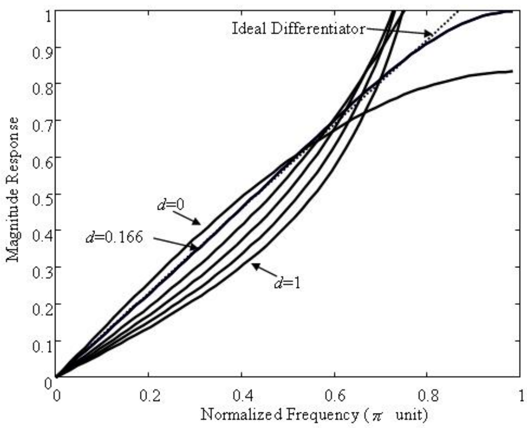

Fig. 2 The amplitude responses with different values of d and the ideal differentiator.

When d is equal to 1, the amplitude response of the system function in (1) becomes infinite at $\Omega = \pi$. If d=0.166, the relative error is less than 1 % (or －40 dB) when $0 \le \omega \le 0.8\pi$. The value of $2/T_d$=0.417 is selected to assume that the maximum value of s in (1) is unity for the normalized frequencies when d=0.166. Under such the examination, the transformation in (1) has a good linearity in amplitude response when d is set equal to 0.166. For d=0.166, we therefore adopt (1) as the system function of the differentiator in discrete-time infinite impulse response format and the selected system function of the first-order differentiator is

$$G_N(z) = 0.417 \cdot \frac{1 - z^{-1}}{1 + 0.166 z^{-1}} \qquad (3)$$

Fig. 3 shows the amplitude responses of $G_N(z)$ compared with linear phase low-pass IIR digital differentiators [4].

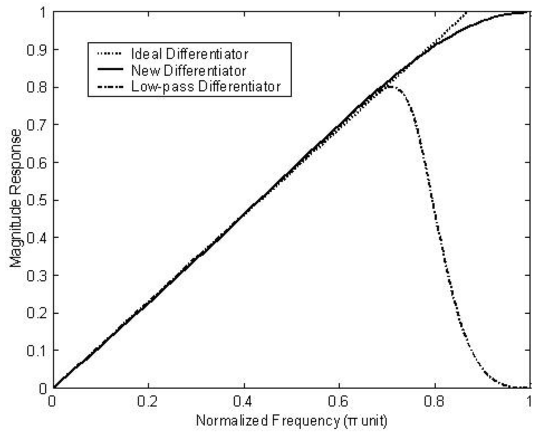

Fig. 3 The amplitude responses of $G_N(z)$, the low-pass IIR digital differentiator and the ideal differentiator.

The relative errors of the differentiators for the corresponding normalized frequency are given in Table I. The relative error of proposed new differentiator is accurate for the operating frequency up to 0.8 within a 1 % error of the normalized frequency.

Table I The relative errors of the differentiators with the corresponding normalized frequency.

Differentiators	Relative errors are less than	Normalized Frequency
New Differentiator	1 % 12 %	0~0.8 π 0.8 π ~ π
Low-pass Differentiator	1 % 100 %	0~0.7 π 0.7 π ~ π

Comparison of the group delays of the new differentiator and the low-pass differentiator are shown in Fig. 4.

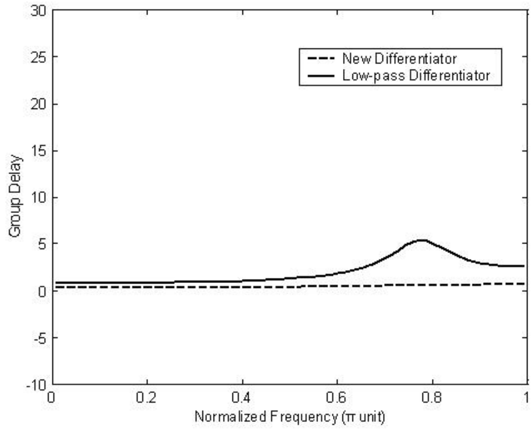

Fig. 4 The group delays of the new differentiator and the low-pass differentiator.

In addition, the group delay of the new differentiator presents constant over the full band. Fig. 5 shows the phase response of the new differentiator, the low-pass differentiator and ideal differentiator. The new differentiator approximately approaches the linear phase. The maximum error of the new differentiator is 9.5° occurring at 0.55 of the normalized frequency.

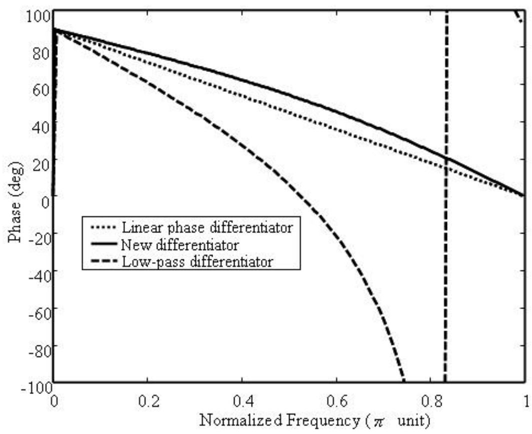

Fig. 5 Phase responses of $G_N(z)$, the low-pass differentiator and the linear phase differentiator.

3.NEW DISCRETE-TIME INTEGRATOR

Many methods have been developed to design integrators using FIR or IIR filters in the study of DSP [10], [11]. To obtain an integrator that better fits the ideal integrator over the entire normalized frequency band, a new integrator is obtained by inverting the transfer function of the new differentiator; the integrator is shown below:

$$H_N(z) = \frac{T_d}{2}\frac{(1+0.166z^{-1})}{(1-z^{-1})} \qquad (4)$$

Fig. 6 shows the amplitude responses of the ideal integrator and (4) for $T_d/2$=0.2 to 1, with 0.2 increments. Table II examines the relative errors of the amplitude responses of (4) with corresponding normalized frequency.

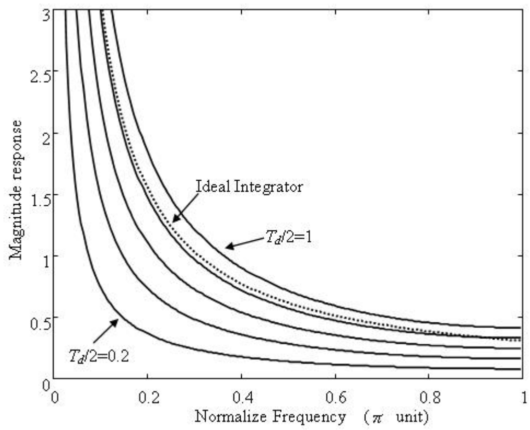

Fig. 6 The amplitude responses of the ideal integrator and $H_N(z)$ for $T_d/2$=0.2 to 1, with 0.2 increments.

Table II The relative errors of the new integrators with the corresponding normalized frequency.

Integrators	Relative errors are less than	Normalized Frequency
$T_d/2$=0.2	77 %	0~ π
$T_d/2$=0.4	54 %	0~ π
$T_d/2$=0.6	31 %	0- π
$T_d/2$=0.8	8 %	0- π
$T_d/2$=1	20% 33%	0-0.8 π 0.8 π - π

In addition, Fig. 7 shows the relative errors of the amplitude responses of the ideal integrator and (4) for $T_d/2$=0.2 to 1, with 0.2 increments. As shown in the figure, when the value of $T_d/2$ is equal to 0.2, the relative error is greater than 70 % for the normalized frequency range is 0-π. If $T_d/2$ is equal to 0.8, the relative error is less than 8 % when the normalized frequency range is 0-π. Therefore, the value of $T_d/2$ is proper to be selected as the system function of a wide-band integrator. The relative error of the amplitude response of (4) is less than 1.1 % when $T_d/2$ is set equal to 0.852 for the normalized frequency range is 0-0.8π. In other words, for $T_d/2$=0.852, we therefore adopt (4) as the system function of the integrator in discrete-time infinite impulse response format and the selected system function of the first-order new integrator is

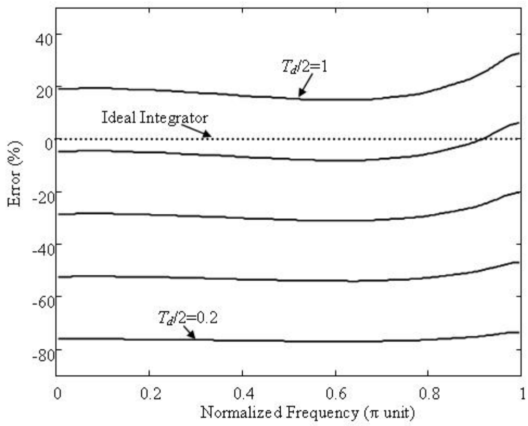

Fig. 7 The relative errors of the amplitude response of the ideal integrator and $H_N(z)$ for $T_d/2$=0.2 to 1, with 0.2 increments.

$$H_N(z) = 0.852\frac{(1+0.166z^{-1})}{(1-z^{-1})} \qquad (5)$$

Fig. 8 shows the magnitude response of the new integrator in (5). The magnitude responses of an ideal, Simpson and Trapezoidal integrators are also shown in Fig. 8. Both Simpson and Trapezoidal integrator are inversely proportional to the normalized frequency and deviate from the value of an ideal integrator in the upper frequency band. The new integrator is approximately the ideal integrator in the normalized frequency band of $0 \le \omega \le \pi$.

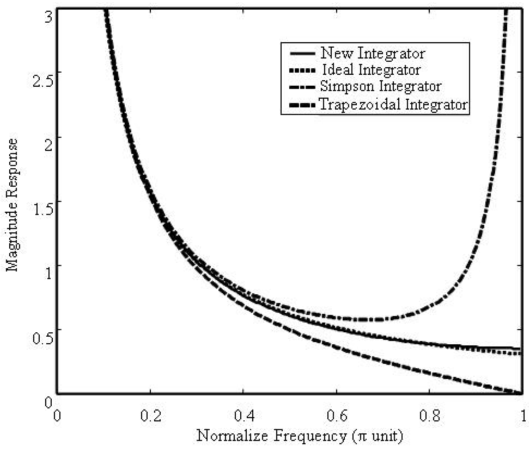

Fig. 8 Magnitude responses of the new integrator, ideal, Simpson and trapezoidal integrators.

Fig. 9 shows the relative error of the magnitude response of the new integrator has an error of less than 1.1 % for the range 0-0.8π of normalized frequency and 6.5 % for the range 0.8π-π of normalized frequency.

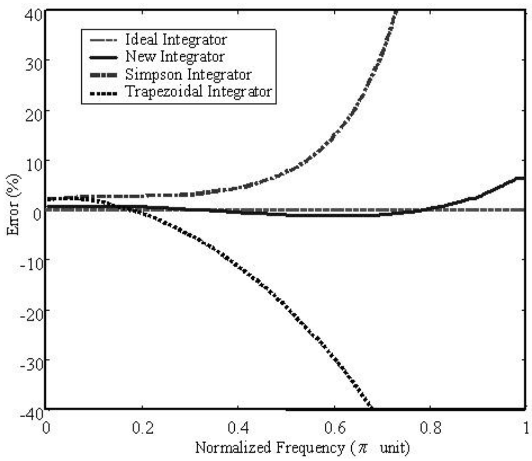

Fig. 9 The relative errors of the magnitude response of the new integrator, Simpson integrator and Trapezoidal integrator.

Fig. 10 shows the phase response of the new integrator which is approximately the linear phase. In addition, the maximum error of the new integrator is 9.5° occurring at 0.55 of the normalized frequency. With the simple formulations, the new integrator gives higher accuracy and greater range of frequencies than traditional integrators.

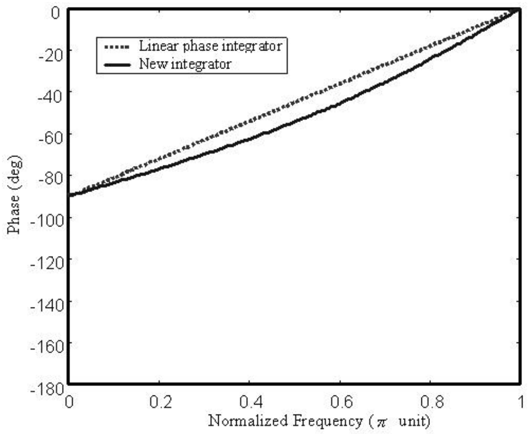

Fig. 10 Phase responses of the new integrator and the linear phase integrator.

4. CONCLUSION

The bilinear transformation is employed to propose both the new differentiator and the integrator. First-order differentiator and integrator as an example are presented. Both the first-order differentiator and integrator have an almost linear phase. The simple and accurate formulations are suitable for real-time applications. The new differentiator and integrator allow them to be eligible for wide-band applications which approximates the ideal differentiator and integrator.

5. REFERENCES

[1] M. I. Skolink, *Introduction to Radar Systems.* New York：Mc-Graw Hill, 1980, pp. 399-408.

[2] R. L. Geiger and G. R. Bailey, "Integrator design for high-frequency active filter applications," *IEEE Trans. Circuits Systems.*, vol. cas-29, no. 9, pp. 595-603, Sept. 1982.

[3] M. A. A1-Alaoui," Novel IIR differentiator from the Simpson rule," *IEEE Trans. Circuits Systems* -I, vol. 41, no. 2, pp. 186-187, Feb. 1994.

[4] M. A. Al-Alaoui, "Linear phase low-pass IIR digital differentiators," *IEEE Trans. Signal Processing,* vol. 55, no. 2, pp. 697-706, Feb. 2007.

[5] C. -C .Tseng, " Design of fractional order digital FIR differentiators," *IEEE Signal Processing Letters*, vol. 8, no. 3, pp. 77-79, March, 2001.

[6] I. R. Khan and R. Ohba, "New design of full-band differentiators based on Taylor series, " *IEEE Proc.-Vis. Image Signal Process*, vol. 146, no. 4, pp. 185-189, Aug. 1999.

[7] M. A. A1-Alaoui, "Novel digital integrator and differentiator," *IEE Electron. Lett.*, vol. 29, no. 4, pp. 376-378, Feb. 1993.

[8] B. Kumar and S.C. Dutta-Roy, "Design of digital differentiators for low-frequencies, " *proc. IEEE*, vol 76, pp. 287-289, 1988.

[9] S. C. Pei and J. J. Shyu, "Analytic closed form matrix for designing high order digital differentiators using eigenapproach, " *IEEE Trans. Signal Processing*, vol. 44, no. 3, pp. 698-701, 1996.

[10] M. A. A1-Alaoui, "A class of second-order integrators and low-pass differentiators," *IEEE Trans. Circuits Systems* -I, vol. 42, no. 4, pp. 220-223, April. 1995.

[11] J. Le Bihan, "Novel class of digital integrators and differentiators," *Electron. Lett.*, vol. 28, no. 15, pp. 1376-1378, 1992.